§2.3 Quantizing the Dirac Field

Anticommutation

We will use the operator formalism of paragraph §1.2 to quantize the Dirac field

Fermions are subject to the Pauli exclusion principle, that is two Fermions with equal quantum numbers cannot occupy the same state

$\longrightarrow$ creation operators anti-commute

Let us start with vacuum $|0\rangle$ and denote by $b_\alpha^\dagger$ the electron creation operator

$\longrightarrow$ $b_\alpha^\dagger |0\rangle$ is state with one electron having quantum numbers α

$\longrightarrow$ two-electron state:

$b_\beta^\dagger b_\alpha^\dagger |0\rangle$

$\rightarrow$ must have $\{b_\alpha^\dagger, b_\beta^\dagger\} = b_\alpha^\dagger b_\beta^\dagger + b_\beta^\dagger b_\alpha^\dagger = 0$

Hermitian conjugation gives: $\{b_\alpha, b_\beta\} = 0$

in particular: $b_\alpha^\dagger b_\alpha^\dagger = 0$

Now add relation

$$\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}$$

Defining operator

$$N := \sum_\alpha b_\alpha^\dagger b_\alpha ,$$

we see

$$\left[\sum_\alpha b_\alpha^\dagger b_\alpha , b_\beta^\dagger\right] = \sum_\alpha [b_\alpha^\dagger b_\alpha , b_\beta^\dagger]$$

use

$$[AB, C] = A\{B, C\} - \{A, C\}B$$

$$\overset{!}{=} \sum_\alpha b_\alpha^\dagger \underbrace{\{b_\alpha, b_\beta^\dagger\}}_{=\delta_{\alpha\beta}} - \underbrace{\{b_\alpha^\dagger, b_\beta^\dagger\}}_{=0} b_\alpha$$

$$= + b_\beta^\dagger$$

$\rightarrow$ N is the "number-operator" as anticipated

check: $N \, b_{\beta_1}^\dagger b_{\beta_2}^\dagger \cdots b_{\beta_K}^\dagger |0\rangle = K \, b_{\beta_1}^\dagger \cdots b_{\beta_K}^\dagger |0\rangle$

The Dirac field

Let us now turn to the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad (1)$$

→ momentum conjugate to ψ:

$$\pi_\alpha = \frac{\delta \mathcal{L}}{\delta(\partial_t \psi_\alpha)} = i\psi_\alpha^\dagger$$

→ anticipate to impose

$$\{\psi_\alpha(\vec{x},t), \psi_\beta^\dagger(\vec{0},t)\} = \delta^{(3)}(\vec{x})\,\delta_{\alpha\beta} \qquad (2)$$

Let's see how this comes about:

$$(i\partial\!\!\!/ - m)\psi = 0$$

→ plugging in plane waves $u(p,s)e^{-ip\cdot x}$ and $v(p,s)e^{ip\cdot x}$ for ψ, we have

$$(p\!\!\!/ - m)u(p,s) = 0 \qquad (3)$$

and

$$(p\!\!\!/ + m)v(p,s) = 0 \qquad (4)$$

the index $s = \pm 1$ is to keep track of the two solutions for each of the above equations: spin up, spin down define $\bar{u} := u^\dagger \gamma^0$, $\bar{v} := v^\dagger \gamma^0$

$\longrightarrow$ $\bar{u}u$, $\bar{v}v$ are Lorentz scalars

In rest frame, eqs. (3) and (4) are

$$(\gamma^0 - \mathbb{1})u = 0, \quad (\gamma^0 + \mathbb{1})v = 0$$

$\to$ in Dirac basis, $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$,

we get

$$u(s=+1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u(s=-1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v(s=+1) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v(s=-1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\to$ $\bar{u}(p,s)u(p,s) = 1, \quad \bar{v}(p,s)v(p,s) = -1 \qquad (5)$

Note that these normalizations are invariant under Lorentz trfs. ($\bar{u}u$, $\bar{v}v$ are scalars!)

Clearly, we also have the orthogonality conditions $\bar{u}v = 0, \quad \bar{v}u = 0 \qquad (6)$

again, Lorentz invariance $\to$ relations (5) and (6) hold in general

In the rest frame

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} = \frac{1}{2}\left(\gamma^0+\mathbb{1}\right)_{\alpha\beta}$$

and

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}_{\alpha\beta} = \frac{1}{2}\left(\gamma^0-\mathbb{1}\right)_{\alpha\beta}$$

In general frames

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \quad (7)$$

and

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} \quad (8)$$

Proof:

Under Lorentz trfs. we have:

$$u(p,s) = u\Big(\Lambda_p \underbrace{\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{=:\,P_{rest}}, s\Big) = U(\Lambda_p)\, u(s)$$

u-vector in rest frame

$$\bar{u}(p,s) = \bar{u}(s)\, U(\Lambda_p)^{-1}$$

$$\longrightarrow \sum_s u(p,s)\,\bar{u}(p,s) = U(\Lambda_p)\Big(\sum_s u(s)\,\bar{u}(s)\Big)U(\Lambda_p)^{-1}$$

$$= U(\Lambda_p)\frac{1}{2}\big(\underbrace{\gamma^0}_{=\frac{1}{m}(P_{rest})_\mu\gamma^\mu}+\mathbb{1}\big)U(\Lambda_p)^{-1} = \frac{1}{2m}\Big(\underbrace{(P_{rest})_\mu\,\Lambda(p)^{\mu\nu}}_{=p^\nu}\gamma_\nu+m\Big)$$

$$= \frac{1}{2m}(\not{p}+m)$$, analogously for $v\bar{v}$ $\square$

Let us now promote $\psi(x)$ to an operator:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\left(\frac{E_p}{m}\right)^{\frac{1}{2}}} \sum_s \left[b(p,s)\,u(p,s)\,e^{-ip.x} + d^\dagger(p,s)\,v(p,s)\,e^{ip.x} \right] \qquad (9)$$

where $E_p = p_0 = +\sqrt{\vec{p}^{\,2} + m^2}$

satisfies Dirac eq.

$\rightarrow$ have to check commutation rel. (2)

$b(p,s)$: annihilation operator for electron with charge $e = -|e|$

$d^\dagger(p,s)$: creation operator for positron with charge $-e = +|e|$

The doubling of particles is needed due to charge conservation:

$$[\underbrace{Q}_{\text{charge operator}}, \underbrace{\mathcal{H}(x)}_{\text{Hamiltonian density}}] = 0$$

Since $\mathcal{H}$ is a polynomial in ψ and $\psi^\dagger$, we have :

$$[Q, \mathcal{H}] = [Q, \mathcal{V}] + [Q, \mathcal{V}^\dagger]$$

we know: $[Q, b] = -eb$, $[Q, b^\dagger] = +eb^\dagger$

thus if $\mathcal{V}$ is composed of only b and $b^\dagger$, we get interaction terms bb and $b^\dagger b^\dagger$ which do not conserve charge!

→ hence we need d with

$$[Q, d] = +ed, \ [Q, d^\dagger] = -ed^\dagger$$

→ $[Q, \mathcal{V}] = -e\mathcal{V}$, $[Q, \mathcal{V}^\dagger] = +e\mathcal{V}^\dagger$

giving $[Q, \mathcal{H}] = 0$

Have commutation relations:

$$\{b(p,s), b^\dagger(p',s')\} = \delta^{(3)}(\vec{p} - \vec{p}')\delta_{ss'}$$

$$\{b(p,s), b(p',s')\} = 0$$

$$\{b^\dagger(p,s), b^\dagger(p',s')\} = 0$$

and similarly for $d, d^\dagger$.

Note: $b, b^\dagger$ anticommute with both $d, d^\dagger$

Proof of eq. (2):

Use that

$$\bar{\psi}(0) = \int \frac{d^3p'}{(2\pi)^{\frac{3}{2}} \left(\frac{E_{p'}}{m}\right)^{\frac{1}{2}}} \sum_{s'} \left[b^\dagger(p',s')\bar{u}(p',s') + d(p',s')\bar{v}(p',s') \right]$$

Now compute

$$\{\psi(\vec{x},0), \bar{\psi}(0)\}$$

$$= \int \frac{d^3p}{(2\pi)^3 \, E_p/m} \sum_s \left[u(p,s)\bar{u}(p,s)e^{-i\vec{p}\cdot\vec{x}} + v(p,s)\bar{v}(p,s)e^{i\vec{p}\cdot\vec{x}} \right]$$

Using equations (7), (8), we get

$$\{\psi(\vec{x},0), \bar{\psi}(0)\} = \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[(\not{p}+m)e^{-i\vec{p}\cdot\vec{x}} + (\not{p}-m)e^{i\vec{p}\cdot\vec{x}} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3(2E_p)} \, 2p^0\gamma^0 e^{-i\vec{p}\cdot\vec{x}} = \gamma^0 \delta^{(3)}(\vec{x})$$

which gives (2) upon multiplication by γ^0 from the right! $\square$

Similarly, we get $\{\psi, \psi\} = 0$, $\{\psi^\dagger, \psi^\dagger\} = 0$

Energy of the vacuum:

$$\mathcal{H} = \pi \frac{\partial \psi}{\partial t} - \mathcal{L} \qquad \text{(Legendre trf.)}$$
$$= \bar{\psi}(i\vec{\gamma}\cdot\vec{\partial} + m)\psi$$

$\longrightarrow$ inserting (9) into this gives

$$H = \int d^3x\, \mathcal{H}(x) \qquad \text{Dirac eq.}$$
$$= \int d^3x\, \bar{\psi}(i\vec{\gamma}\cdot\vec{\partial} + m)\psi \overset{\downarrow}{=} \int d^3x\, \bar{\psi} i\gamma^0 \frac{\partial \psi}{\partial t}$$

finally giving

$$H = \int d^3p \sum_s E_p \left[b^\dagger(p,s) b(p,s) - d(p,s) d^\dagger(p,s) \right]$$

(exercise)

We anticommute

$$-d(p,s)d^\dagger(p,s) = d^\dagger(p,s)d(p,s) - \delta^{(3)}(\vec{0})$$

to get

$$H = \int d^3p \sum_s E_p \left[b^\dagger(p,s) b(p,s) + d^\dagger(p,s) d(p,s) \right] \underbrace{- \delta^{(3)}(\vec{0}) \int d^3p \sum_s E_p}_{}$$

"vacuum energy" negative!

$$= -\frac{1}{(2\pi)^3} \int d^3x \int d^3p \sum_s 2\left(\tfrac{1}{2}E_p\right)$$

positron + electron